

On the time consistency of collective preferences

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Abstract

A model of collective decisions made by a finite number of agents with constant but heterogeneous discount factors is developed. Collective utility is obtained as the weighted sum of all individual utilities with time-varying weights. It is shown that under standard separability assumptions, collective preferences may be nonstationary but still satisfy time consistency.

Key words: Dynamic collective choice; Time consistency; Nonstationary preferences; Recursive utility; Pareto optimality

JEL classification: D71, D90, D61, C61

AMS subject classification: 37N40, 91B10, 91B69, 49L20, 49K35

1 Introduction

This paper develops a model of collective dynamic decisions made by time consistent agents with different discount factors. One of the main issues addressed in this framework is whether the aggregation of heterogeneous time preferences necessarily implies time inconsistency. This kind of aggregation problem has been long recognized, but it still poses challenges. For instance, Dumas [7] shows that difficulties in modelling aggregate behavior in the presence of heterogeneity may arise even in a two-agent setting with homogeneous discount factors. The study by Gollier and Zeckhauser [11] takes a more general approach and analyzes an exchange economy composed of a finite group of consumers with constant but heterogeneous discount factors. They show that Paretian aggregation of individual preferences does not yield in general a constant aggregate discount rate. Optimality implies that individual shares of a common consumption stream change over time, which allows to define an aggregate discount rate. Collective impatience is a weighted average of individual discount rates, with weights being proportional to individual *tolerance for consumption fluctuations* (a concept similar to “risk tolerance” when agents have preferences over stochastic consumption streams). Pareto optimality also dictates that the more

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impatient members get a higher share earlier in time, and the more tolerant agents bear a larger share of aggregate fluctuations.

The conditions for time consistent aggregation of individual preferences are very strong, as shown by Zuber [21]. Aggregate preferences are stationary and Pareto optimal if and only if individual and collective preferences are additively separable with exponential discounting, and all agents have the same discount rate. There are also strong implications for collective choice models. Recent work by Jackson and Yariv [13, 14] shows that when preferences of heterogeneous agents evaluating a stream of common consumption are aggregated via a collective utility function that satisfies Pareto optimality, these collective preferences must be dictatorial or time inconsistent. On the other hand, if aggregation rules are Paretian and nondictatorial, they exhibit a specific form of time inconsistency, known as *present-bias* or *decreasing impatience*. Actually, the authors perform an experimental study and find that this is one of the main sources of time inconsistent choices.

All the studies cited above share a common assumption: aggregate utility is a weighted sum of all individual utilities with utility weights that are *constant over time*. It is shown below that the latter is key for their results, since relaxing this assumption makes time consistent aggregation of individuals with heterogeneous discount factors possible. To do this, the problem of constructing Pareto optimal allocations with heterogeneous consumers is extended to allow for time-varying utility weights, based on the framework developed by Lucas and Stokey [17]. Utility weights then become a state variable in the dynamic program. Important extensions introduced by Dana and Le Van [5, 6], such as allowing for unbounded returns and more general specifications of the discounting process, are also included in the analysis.

A sharp insight provided by Halevy [12] will also be used to characterize the collective preferences obtained in this work. Specifically, assume that preferences are represented by a sequence of preference relations $\{\succsim_t\}_{t=0}^\infty$ defined over temporal payments $(c, t) \in \mathbb{R}_+ \times \mathbb{Z}_+$. Then, for every $t, t' \geq 0$, $b, c \in \mathbb{R}_+$, and $\tau, \tau' \geq 0$, three distinct properties are defined by the following relations

$$(A1) \text{ Stationarity: } (b, t + \tau) \sim_t (c, t + \tau') \iff (b, t' + \tau) \sim_t (c, t' + \tau'),$$

$$(A2) \text{ Time invariance: } (b, t + \tau) \sim_t (c, t + \tau') \iff (b, t' + \tau) \sim_{t'} (c, t' + \tau'),$$

$$(A3) \text{ Time consistency: } (b, t + \tau) \sim_t (c, t + \tau') \iff (b, t + \tau) \sim_{t'} (c, t + \tau').$$

It is shown in [12] that any two properties imply the third. Hence, if preferences are time invariant, they satisfy time consistency if and only if they are stationary. But the distinction between stationarity and time consistency as two separate phenomena has been often overlooked in the literature.

To summarize, this paper contains two main contributions. First, it develops a framework to construct collective preferences from the Pareto optimal consumption and saving decisions of time consistent agents with heterogeneous discount factors. The other main contribution is that it provides sufficient conditions for these collective preferences to satisfy time consistency.

2 Setup and Aggregation

2.1 Generalities

Let \mathbb{R} be the set of all real numbers. Then, $\mathbb{R}_+ := [0, +\infty)$ and $\underline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. The set of all nonnegative integers is \mathbb{Z}_+ . Vectors in the n -dimensional Euclidean space \mathbb{R}_+^n , the n -fold cartesian product of \mathbb{R}_+ , are denoted by x . For infinite sequences of these n -dimensional vectors, the notation used is $\mathbf{x} := (x_0, x_1, \dots)$. The space of real-valued sequences ℓ^∞ is endowed with the sup norm, $\|\mathbf{x}\|_\infty := \sup_t |x_t|$. If $n > 1$ and each component is in ℓ^∞ , the corresponding product space will be denoted as $(\ell^\infty)^n$ (for ease of notation, the infinity superscript will be dropped if no ambiguity arises).

The following conventions, which are commonly used in convex optimization problems, are adopted (see [19]),

$$(C1) \quad 0 \cdot (\pm\infty) = (\pm\infty) \cdot 0 = 0;$$

$$(C2) \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty, \quad \text{for all } a \in (0, +\infty];$$

$$(C3) \quad a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty, \quad \text{for all } a \in [-\infty, 0).$$

For some integer $n \geq 2$, a set $N = \{1, \dots, n\}$ of infinitely lived agents make collective consumption and savings decisions. Time is discrete and denoted by $t = 0, 1, 2, \dots$. There is a single consumption good which can be either consumed or transformed one-to-one in capital available next period. Production occurs in a single unit where agents pool their capital holdings and obtain a certain deterministic return in terms of the consumption good, which is called “consumption” for simplicity. But the framework is quite general and admits various interpretations.

Time preferences are represented by an additively separable intertemporal utility function with instantaneous utility u , and constant geometric discounting. Instantaneous utility is assumed to be common to all agents, but discount factors δ^i are heterogeneous and satisfy $1 > \delta^1 > \dots > \delta^n > 0$. Therefore, each agent i assigns an utility value (in period $t = 0$) to a consumption path $\mathbf{x}^i := \{x_t^i\}_{t=0}^\infty$ in the space ℓ_+ of non-negative bounded sequences

$$(2.1) \quad w_0^i(\mathbf{x}^i) := \sum_{t=0}^{\infty} (\delta^i)^t u(x_t^i), \quad i = 1, \dots, n.$$

The following assumptions on the primitives of the problem, preferences and technology, will be used throughout the paper.

(U1) The instantaneous utility function $u : \mathbb{R}_+ \rightarrow \underline{\mathbb{R}}$ is continuous, strictly increasing, and strictly concave. At the origin, either $u(0) = 0$ or $u(0) = -\infty$.

(U2) The instantaneous utility function u is twice continuously differentiable on $\mathbb{R}_+ \setminus \{0\}$. If $u(0) = 0$, then $\lim_{x \rightarrow 0^+} u'(x) = +\infty$.

Technology is given by a standard neoclassical production function that transforms aggregate capital into output.

- (T1) The production function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, strictly increasing, strictly concave, and $f(0) = 0$.
- (T2) The production function f is differentiable on $\mathbb{R}_+ \setminus \{0\}$, $\lim_{k \rightarrow 0^+} f'(k) > (1/\delta^n)$, and $\lim_{k \rightarrow +\infty} f'(k) = 0$.

Remark 2.1. Note that these assumptions on technology imply the existence of a maximum level of sustainable capital $k_m > 0$, so there is no loss of generality in restricting the state space to the closed interval $K := [0, k_m]$. This implies that the set of feasible consumption (and utilities) is bounded above. Moreover, (U2)–(T2) guarantee that equilibrium paths are interior.

Definition 2.2. An *allocation* is a pair (\hat{x}_t, k_{t+1}) for every t , which consists on a consumption profile $\hat{x}_t := (x_t^1, \dots, x_t^n) \in \mathbb{R}_+^n$ and aggregate capital for the next period $k_{t+1} \in \mathbb{R}_+$. Given $k_0 \geq 0$, a capital path $\mathbf{k} := \{k_{t+1}\}_{t=0}^\infty$ is said to be *feasible* if each element of the sequence belongs to the set

$$(2.2) \quad \Gamma(k_t) := \{k_{t+1} \in K : 0 \leq k_{t+1} \leq f(k_t)\},$$

which is usually called the *feasible correspondence*.

Definition 2.3. The set of all *feasible capital paths* from k_0 is defined as

$$(2.3) \quad \Pi(k_0) := \{\mathbf{k} \in \ell_+ : k_{t+1} \in \Gamma(k_t), t = 0, 1, \dots; k_0 \text{ given}\}.$$

Definition 2.4. The set of all *feasible consumption paths* from k_0 is defined as

$$(2.4) \quad \Omega(k_0) := \{\hat{\mathbf{x}} \in \ell_+^n : 0 \leq \sum_i x_t^i \leq f(k_t) - k_{t+1}, t = 0, 1, \dots; \text{ for some } \mathbf{k} \in \Pi(k_0)\}.$$

2.2 A Pareto problem with heterogeneous discounting

The methodology for constructing the Pareto optimal allocations is mainly based on [17] and the extensions made in [5, 6]. Although an important difference is that individual preferences in the current framework are a primitive of the problem, instead of being derived from a time aggregator.

The Pareto or utility weights for each agent are allowed to vary over time. For each period t , the set of weights is given by a vector $\theta_t := (\theta_t^1, \dots, \theta_t^n)$ in the n -dimensional simplex, denoted by Θ^n and defined as

$$(2.5) \quad \Theta^n := \{\theta \in \mathbb{R}_+^n : \theta^i \geq 0, i = 1, \dots, n; \text{ and } \sum_i \theta^i = 1\}.$$

The utility possibility set $\mathcal{U}(k)$ contains the possible combinations of utility available to the n agents when the initial capital stock is $k \geq 0$,

$$\mathcal{U}(k) := \{z \in \mathbb{R}^n : z^i = w^i(\mathbf{x}^i), \ i = 1, \dots, n, \text{ for some } \hat{\mathbf{x}} \in \Omega(k)\},$$

and the value function of the Pareto problem is defined as the support function of this set, that is,

$$V(k, \theta) := \sup_{z \in \mathcal{U}(k)} \sum_{i=1}^n \theta^i z^i.$$

A characterization of the value function is given in the following proposition. For detailed proofs, see [5] and [17].¹

Proposition 2.5. *The value function $V : K \times \Theta^n \rightarrow \mathbb{R}$ satisfies the following properties:*

- (a) V is continuous on $K \times \Theta^n$;
- (b) For each $\theta \in \Theta^n$, $V(k, \cdot) : K \rightarrow \mathbb{R}$ is strictly increasing and strictly concave;
- (c) For each $k \in K$, $V(\cdot, \theta) : \Theta^n \rightarrow \mathbb{R}$ is homogeneous of degree one and strictly convex;
- (d) V is continuously differentiable in the interior of $K \times \Theta^n$.

Let $x := \sum_i x^i$ denote aggregate consumption and $X \subset \mathbb{R}_+$ the aggregate consumption space. It is convenient to define the set of all feasible aggregate consumption paths from k_0 separately from Ω , therefore

$$(2.6) \quad X(k_0) := \{\mathbf{x} \in \ell_+ : 0 \leq x_t \leq f(k_t) - k_{t+1}, \ t = 0, 1, \dots; \text{ for some } \mathbf{k} \in \Pi(k_0)\}.$$

Next result, which completely characterizes Pareto optimal allocations, summarizes Propositions 3.1–3.2 and Theorem 3.1 in [5], and is stated below for convenience.

Proposition 2.6. *Under assumptions (U1)–(U2) on preferences and (T1)–(T2) on technology, an allocation $(\hat{\mathbf{x}}, \mathbf{k}) \in \Omega(k_0) \times \Pi(k_0)$ is Pareto optimal if and only if there exist \mathbf{k}, \mathbf{z} such that $k_{t+1} \in \Gamma(k_t)$, $x_t \in X(k_t)$, and $z_t \in \tilde{\mathcal{U}}(k_t)$ for all t , where $\tilde{\mathcal{U}}$ is the frontier of \mathcal{U} , and $w_t^i = u(x_t^i) + \delta^i z_{t+1}^i$ for all $i \in N$ and for all t . Moreover, \mathbf{k} and \mathbf{z} are uniquely determined.²*

In light of the above, the Pareto problem can be formulated as follows. For each (k, θ) in $K \times \Theta^n$, a nonnegative consumption profile $\hat{x} := (x^1, \dots, x^n)$, next-period capital y , continuation utilities z , and next-period Pareto weights τ are chosen to solve the following

¹See also [9] for some results that are particularly useful for the case where returns are unbounded.

²If S is a closed set, and \bar{S} denotes its closure, the *topological frontier* \tilde{S} of S is defined as $\bar{S} \setminus S$.

program:

$$\begin{aligned}
(\text{PP}) \quad & \sup_{\hat{x}, y \geq 0, z \in \mathcal{U}} \inf_{\tau \in \Theta^n} \sum_{i=1}^n \theta^i [u(x^i) + \delta^i z^i], \\
& \text{s.t.} \quad \sum_{i=1}^n x^i + y \leq f(k), \\
& \quad \sum_{i=1}^n \tau^i z^i - V(y, \tau) \leq 0.
\end{aligned}$$

Based on this formulation, the following theorem lays the foundations for the preference aggregation process and subsequent results.

Theorem 2.7. *There exist maps $U : X \times \Theta^n \rightarrow \mathbb{R}$, $\mu : \Theta^n \rightarrow \mathbb{R}_+$, and $F : \Theta^n \rightarrow \Theta^n$, such that the value of the Pareto problem (PP) satisfies the following functional equation*

$$(2.7) \quad V(k, \theta) = \sup_{y \in \Gamma(k)} \left[U(f(k) - y, \theta) + \mu(\theta) V(y, F(\theta)) \right],$$

for all (k, θ) in the interior of $K \times \Theta^n$.

Proof. Fix $k \in (0, k_m)$ and $\theta^i > 0$ for all $i \in N$. Let λ, μ be nonnegative Lagrange multipliers associated to the inequality restrictions in (PP), and form the Lagrangean

$$\begin{aligned}
(2.8) \quad \mathcal{L}(\hat{x}, y, z, \tau, \lambda, \mu | k, \theta) &= \sum_i \theta^i [u(x^i) + \delta^i z^i] + \lambda [f(k) - \sum_i x^i - y] \\
&\quad - \mu [\sum_i \tau^i z^i - V(y, \tau)].
\end{aligned}$$

Denote by $\hat{X} \subset \mathbb{R}_+^n$ the space of \hat{x} allocations, and let $\Phi := \hat{X} \times \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+$ and $\Psi := \Theta^n \times \mathbb{R}_+$. The proof is divided into several steps.

Step 1. Note that Proposition 2.5 implies that (PP) is a saddle-point problem with a concave-convex function defined on products of convex sets that satisfies *strong duality*, hence

$$(2.9) \quad V(k, \theta) = \sup_{(\hat{x}, y, z, \lambda) \in \Phi} \inf_{(\tau, \mu) \in \Psi} \mathcal{L}(\hat{x}, y, z, \tau, \lambda, \mu | k, \theta),$$

and the sup and inf above can be interchanged.³ Also, by Lemma 1 in [16], the supremum over two variables can be split into two suprema. Then, the right-hand side of (2.9) is

³For a recent treatment on strong duality in convex optimization problems, see [2].

equivalent to:

$$(2.10) \quad \sup_{\hat{x} \in \hat{X}} \sup_{(y, z, \lambda) \in \Phi_{\hat{x}}} \inf_{(\tau, \mu) \in \Psi} \left\{ \left[\sum_i \theta^i u(x^i) + \lambda(x - \sum_i x^i) \right] + \lambda[f(k) - x - y] \right. \\ \left. + \sum_i (\theta^i \delta^i - \mu \tau^i) z^i + \mu V(y, \tau) \right\},$$

where $\Phi_{\hat{x}} := \{(y, z, \lambda) \in \mathbb{R}_+ \times \mathcal{U} \times \mathbb{R}_+ : (\hat{x}, y, z, \lambda) \in \Phi\}$.

Step 2. Consider the following auxiliary program

$$\sup_{\hat{x} \in \hat{X}} \left[\sum_i \theta^i u(x^i) + \lambda(x - \sum_i x^i) \right],$$

for some $0 < x \leq f(k)$ given. Since this problem is convex and differentiable, and the solution is interior for each i , Karush-Kuhn-Tucker (KKT) conditions are necessary and sufficient for optimality. Then, there is a map $\tilde{s} : [0, 1] \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$, such that for each $i \in N$

$$(2.11) \quad \theta^i u'(\tilde{s}(\theta^i, \lambda)) = \lambda,$$

and $\sum_i \tilde{s}(\theta^i, \lambda) \leq x$. The value of this auxiliary program is

$$(2.12) \quad Q(\lambda, x, \theta) := \sum_i \theta^i u(\tilde{s}(\theta_i, \lambda)) + \lambda[x - \sum_i \tilde{s}(\theta^i, \lambda)].$$

Next, replace $\sup_{\hat{x}} [\sum_i \theta^i u(x^i) + \lambda(x - \sum_i x^i)]$ with $Q(\lambda, x, \theta)$ in (2.10), and split the sup again, so the problem can be written as

$$\sup_{z \in \mathcal{U}} \sup_{(y, \lambda) \in \Phi_{\hat{x}, z}} \inf_{(\tau, \mu) \in \Psi} \left\{ Q(\lambda, x, \theta) + \lambda[f(k) - x - y] + \left[\sum_i (\theta^i \delta^i - \mu \tau^i) z^i \right] + \mu V(y, \tau) \right\},$$

where $\Phi_{\hat{x}, z} := \{(y, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+ : (y, z, \lambda) \in \Phi_{\hat{x}}\}$.

Step 3. Now, solve the following auxiliary program for $0 \leq y < f(k)$ given,

$$(2.13) \quad \sup_{z \in \mathcal{U}} \inf_{(\tau, \mu) \in \Psi} \left[\sum_i (\theta^i \delta^i - \mu \tau^i) z^i + \mu V(y, \tau) \right].$$

If $\mu = 0$, then each i should get the maximum of z^i on \mathcal{U} , but this is infeasible. Assume $\mu > 0$, which implies $\sum_i \tau^i z^i = V(y, \tau)$ by complementary slackness. Differentiating the objective in (2.13) with respect to τ^i yields $z^i = \partial V / \partial \tau^i$, $i = 1, \dots, n$. On the other hand, given that $0 < \theta^i < 1$, optimality requires that τ^i satisfies

$$(2.14) \quad \theta^i \delta^i = \mu \tau^i,$$

for each $i \in N$. This in turn implies $0 < \tau^i < 1$ for all i , so adding up (2.14) over $i \in N$

yields

$$(2.15) \quad \mu = \sum_{i=1}^n \theta^i \delta^i.$$

At the optimum, μ is a function of the utility weights only. Slightly abusing notation, call this map $\mu : \Theta^n \rightarrow \mathbb{R}_+$. Note that (2.14)–(2.15) defines a transition map $F : \Theta^n \rightarrow \Theta^n$ for the utility weights, so $\tau = F(\theta)$. Taking all this into consideration, the right-hand side of (2.9) can be further simplified and written as

$$(2.16) \quad \sup_{(y, \lambda) \in \Phi_{\hat{x}, z}} \{Q(\lambda, x, \theta) + \lambda[f(k) - x - y] + \mu(\theta)V(y, F(\theta))\}.$$

Step 4. By (2.11), individual allocations depend only on (θ, λ) , so the choice of \hat{x} can be formulated in terms of aggregate consumption x , instead of λ . Given that u is strictly increasing and $\lambda > 0$ holds at the optimum, set $x = \sum_i \tilde{s}(\theta^i, \lambda)$ in (2.12) and use this expression to implicitly define a function $\lambda : X \times \Theta^n \rightarrow \mathbb{R}_+$. This, in turn, allows to define a map $U : X \times \Theta^n \rightarrow \mathbb{R}$, given by

$$U(x, \theta) := Q(\lambda(x, \theta), x, \theta) = \sum_{i=1}^n \theta^i u(\tilde{s}(\theta^i, \lambda(x, \theta))).$$

Finally, substituting the above expression into (2.16) yields

$$V(k, \theta) = \sup_{x, y \geq 0} \left[U(x, \theta) + \mu(\theta)V(y, F(\theta)) : x + y \leq f(k) \right],$$

that is equivalent to (2.9). It is readily verified that $U(x, \cdot)$ is strictly increasing.⁴ This, together with the strict monotonicity of $V(y, \cdot)$, implies that the resource restriction holds with equality which gives (2.7).

Given that (k, θ) was chosen arbitrarily in the interior of $K \times \Theta^n$, the above conclusions can be extended to that set. This completes the proof. \blacksquare

Remark 2.8. Note that the optimal value for the Lagrange multiplier, taken to be a function of θ_t , can be interpreted as a *factor of time preference* for the collective program and is related to the one-period discount rate.⁵ In fact, it is clear from (2.15), that this aggregate factor of time preference is the weighted arithmetic mean of the individual

⁴To see this, implicitly differentiate $\tilde{s}^i := \tilde{s}(\theta^i, \lambda)$ with respect to λ in (2.11) to obtain $\partial \tilde{s}^i / \partial \lambda = (\theta^i u''(\tilde{s}(\theta^i, \lambda)))^{-1} < 0$, for each i . It also follows from implicit differentiation that $\partial \lambda / \partial x = \sum_i \partial \tilde{s}^i / \partial \lambda < 0$. Therefore,

$$\frac{\partial U}{\partial x} = \sum_{i=1}^n \theta^i u'(\tilde{s}^i) \frac{\partial \tilde{s}^i}{\partial \lambda} \frac{\partial \lambda}{\partial x} > 0.$$

⁵The *instantaneous rate of time preference* is defined as $\rho(\cdot) := 1/\mu(\cdot) - 1$.

discount factors, with weights given by $\theta_t \in \Theta^n$. The evolution of $\mu(\theta_t)$ over time is entirely explained by the evolution of the utility weights.

Remark 2.9. A direct consequence of Theorem 2.7 is that collective preferences have a representation of the form

$$(2.17) \quad W_t = U(x_t, \theta_t) + \mu(\theta_t)W_{t+1}, \quad t = 0, 1, \dots,$$

where $W_t := \sum_i \theta_t^i w_t^i$ is an aggregate utility index, the collective instantaneous utility function $U(x_t, \theta_t)$ is constructed by aggregating individual *indirect utility functions*, and $\mu(\theta_t)$ is the aggregate discount rate. Then, (2.17) defines recursively an intertemporal utility function that assigns to each infinite stream of aggregate consumption and Pareto weights $(\mathbf{x}, \boldsymbol{\theta})$ in $\ell_+ \times (\Theta^n)^\infty$ an utility measure for the collective formed by the members of N . This representation of collective preferences also defines an aggregate *discount factor*

$$\beta_t := \prod_{s=0}^{t-1} \mu(\theta_s), \quad t = 0, 1, \dots$$

It may seem odd having collective preferences that depend, directly or indirectly, on a vector of utility weights θ_t , but there are a few antecedents in the literature. One way in which the recursive representation (2.17) can be rationalized is using the concept of *variational utility* introduced by Geoffard [10]. The variational utility of a consumption path is defined as the minimum value of an additive criterion, taken over all possible future discount rates. Note the close relation between this idea and the formulation of the Pareto problem given in (2.10). The variational utility approach includes time additive and recursive preferences as special cases. Henceforth, the model developed in this paper can be understood in the context of a broader class of deterministic models where optimal allocations can be characterized as the solution of a dynamic program involving one value function representing aggregate utility. A stochastic formulation of this approach can be found in the work of Dumas, Uppal, and Wang [8].

Remark 2.10. The dynamic program (PP) admits a sequential formulation with aggregate recursive preferences, which will be called the recursive preference formulation (RPF), and has the form

$$\begin{aligned} \text{(RPF)} \quad & \sup_{\mathbf{x} \in X(k_0), \mathbf{k} \in \Pi(k_0)} \sum_{t=0}^{\infty} \beta_t U(x_t, \theta_t) \\ & \text{s.t.} \quad x_t + k_{t+1} \leq f(k_t), \quad t = 0, 1, \dots, \\ & \quad \beta_{t+1} \leq \mu(\theta_t) \beta_t, \quad t = 0, 1, \dots, \\ & \quad \theta_{t+1} = F(\theta_t), \quad t = 0, 1, \dots, \\ & \quad k_0, \theta_0 > 0 \text{ given, } \beta_0 = 1. \end{aligned}$$

Furthermore, the value function V associated with (RPF) satisfies a generalized Bellman equation given by (2.7).

From (2.14) and (2.15), the dynamics of utility weights are determined by the following updating process

$$(2.18) \quad \theta_{t+1}^i = \frac{\theta_t^i \delta^i}{\sum_j \theta_t^j \delta^j}, \quad i = 1, \dots, n; \quad t = 0, 1, \dots$$

Note that the process is quite relentless: if $\theta_t^i = 0$ for some i at any period t , then it will take a zero value forever, and agent i is practically removed from the group. In this paper, this possibility is ruled out in equilibrium by assumptions (U2) and (T2), as long as $\theta_0^i > 0$ for all $i \in N$.

Another consequence of the dynamics derived from (2.18) is that for any $j = 2, \dots, n$, it follows that

$$\frac{\theta_{t+1}^j}{\theta_{t+1}^1} = \left(\frac{\delta^j}{\delta^1} \right) \frac{\theta_t^j}{\theta_t^1} \rightarrow 0, \quad \text{as } t \rightarrow +\infty,$$

so the relative weight on aggregate utility vanishes asymptotically for all agents, with the exception of the most patient. This and (2.5) immediately imply that $\theta_t^j \rightarrow 0^+$, for all $j = 2, \dots, n$, and $\theta_t^1 \rightarrow 1^-$ as t approaches infinity. Therefore, agent 1's consumption equals x_t , while the others consume zero (or a minimum subsistence level), a result known as *Ramsey's conjecture*. However, the analysis focuses on the case where $\theta_t^i > 0$ holds for every $i \in N$ and for any finite t , so the equilibrium implied by Ramsey's conjecture holds only asymptotically.

3 Collective Preferences

This section develops sufficient conditions under which the optimal allocations x_t^i , for each $i \in N$ and for each t , can be written as a function of the state of the system (x_t, θ_t) , and the collective utility $U(x_t, \theta_t)$ is multiplicatively separable (up to an additive constant). This is explained by the fact that the literature on dynamic choice generally assumes a preference relation \succsim_t over dated commodities (x, t) that can be represented by a separable function, e.g., where the discount factor, say $D(t)$, depends only on time, and instantaneous utility $U(x)$ on the level of consumption (or “reward” using terminology from behavioral economics).

In order to characterize the problem, it is useful to define an index of absolute *tolerance for consumption fluctuations* (TCF), introduced in [11], which is related to the curvature of the instantaneous utility function. Using a specific functional form for individual instantaneous utility u , that yields a linear TCF index, allows to obtain the desired separability for U . This is hardly surprising, given the analogous relation between the concavity of the instantaneous utility function and risk aversion/TCF.

3.1 Separability

Time additivity, as shown in the proof of Theorem 2.7, has an important consequence on Pareto optimal allocations, namely that the allocation of aggregate consumption among agents can be solved independently from the collective investment decision. Hence, it is possible to treat the distributive problem as an *intratemporal* or static program. This fact, a consequence of Fisher's separation theorem, is key to obtain the separable form for U .

Fix $t \geq 0$ together with a feasible level of aggregate consumption $x_t > 0$. Given $\theta_0 > 0$, the vector of utility weights at any period t is given by $\theta_t = F^t(\theta_0)$, where F^t is the t -fold application of the map F described in (2.18). Then, the collective instantaneous utility function $U : X \times \Theta^n \rightarrow \underline{\mathbb{R}}$ is the value of the following program

$$(S) \quad \begin{aligned} U(x_t, \theta_t) &= \sup_{\hat{x}_t \in \hat{X}} \sum_{i=1}^n \theta_t^i u(x_t^i) \\ \text{s.t.} \quad &\sum_{i=1}^n x_t^i \leq x_t. \end{aligned}$$

Definition 3.1. A *sharing rule* is a map $s : X \times \Theta^n \rightarrow \mathbb{R}_+^n$ that satisfies

$$(3.1) \quad \sum_{i=1}^n s^i(x_t, \theta_t) = x_t.$$

A *Pareto optimal sharing rule* is a sharing rule that solves (S).

The set \mathcal{S} of all sharing rules s is nonempty, since the allocation $s^i = x_t/n$ for all $i \in N$ is always feasible. Note that there is a correspondence between this sharing rule and the one defined in (2.11), given by $s^i(x, \theta) = \tilde{s}(\theta^i, \lambda(x, \theta))$. The following result characterizes aggregate utility and the optimal sharing rule.

Proposition 3.2. *The aggregate instantaneous utility function $U : X \times \Theta^n \rightarrow \underline{\mathbb{R}}$ and the Pareto optimal sharing rule $s : X \times \Theta^n \rightarrow \mathbb{R}_+^n$ satisfy the following properties:*

- (a) *For each $\theta \in \Theta^n$, $U(x, \cdot) : X \rightarrow \mathbb{R}$ is strictly increasing and strictly concave;*
- (b) *For each $x \in X$, $U(\cdot, \theta) : \Theta^n \rightarrow \mathbb{R}$ is homogeneous of degree one;*
- (c) *U is twice continuously differentiable in the interior of $X \times \Theta^n$;*
- (d) *For each $i \in N$: $s^i(0, \theta) = 0$, for all $\theta \in \Theta^n$; and $s^i(\cdot, \theta) : \Theta^n \rightarrow \mathbb{R}_+^n$ is homogeneous of degree zero, for all $x \in X$;*
- (e) *s is continuously differentiable in the interior of $X \times \Theta^n$.*

Proof. The proofs for parts (a), (c), and (e) are omitted and can be found in [18]. In order to simplify the proofs for the remaining results, it will be assumed that the differentiability of U and s has already been established.

Fix $x \in (0, +\infty)$ and suppose that $\theta^i > 0$ holds for all $i \in N$. Differentiating (3.1) with respect to θ^i yields

$$(3.2) \quad \sum_{j=1}^n \frac{\partial s^j(x, \theta)}{\partial \theta^i} = 0.$$

Next, differentiate $U(x, \theta) = \sum_i \theta^i u(s^i(x, \theta))$ with respect to θ^i , so that

$$\begin{aligned} \frac{\partial U(x, \theta)}{\partial \theta^i} &= u(s^i(x, \theta)) + \sum_{j=1}^n \theta^j u'(s^j(x, \theta)) \frac{\partial s^j(x, \theta)}{\partial \theta^i} \\ &= u(s^i(x, \theta)) + \lambda(x, \theta) \sum_{j=1}^n \frac{\partial s^j(x, \theta)}{\partial \theta^i}, \end{aligned}$$

where the second equality uses the fact that $\theta^i u'(s^i(x, \theta)) = \lambda(x, \theta)$ from the KKT optimality conditions. By (3.2), the above equality reduces to

$$(3.3) \quad \frac{\partial U(x, \theta)}{\partial \theta^i} = u(s^i(x, \theta)), \quad \text{for all } i \in N,$$

which in turn implies that

$$\sum_{i=1}^n \theta^i \frac{\partial U(x, \theta)}{\partial \theta^i} = \sum_{i=1}^n \theta^i u(s^i(x, \theta)) = U(x, \theta),$$

hence $U(\cdot, \theta)$ is homogeneous of degree one by Euler's theorem for homogeneous functions. This proves (b).

For part (d), since s^i is nonnegative for each $i \in N$, it is clear from (3.1) that $x = 0$ must imply $s^i(0, \theta) = 0$ for all $\theta \in \Theta^n$. To prove that $s^i(\cdot, \theta)$ is homogeneous of degree zero, by a corollary from Euler's theorem for homogeneous functions, it is straightforward to show that if $U(\cdot, \theta)$ is homogeneous of degree one, its partial derivatives are homogeneous of degree zero. Differentiating (3.3) with respect to θ^j , $j = 1, \dots, n$, gives

$$\frac{\partial}{\partial \theta^j} \left[\frac{\partial U(x, \theta)}{\partial \theta^i} \right] = u'(s^i(x, \theta)) \frac{\partial s^i(x, \theta)}{\partial \theta^j},$$

therefore

$$\sum_{j=1}^n \theta^j \frac{\partial}{\partial \theta^j} \left[\frac{\partial U(x, \theta)}{\partial \theta^i} \right] = \frac{\lambda(x, \theta)}{\theta^i} \left[\sum_{j=1}^n \theta^j \frac{\partial s^i(x, \theta)}{\partial \theta^j} \right] = 0.$$

But $\lambda(x, \theta)$ is positive in any interior equilibrium and $\theta^i > 0$ by hypothesis, so the sum in square brackets in the second equality above must vanish, yielding the desired result. ■

Remark 3.3. Intuitively, note that the homogeneity of $U(\cdot, \theta)$ can also be obtained from the dynamic problem. Given that $V(\cdot, \theta)$ and $\mu(\theta)$ are homogeneous of degree one, and $F(\theta)$ is homogeneous of degree zero, it follows from the Bellman equation in (2.16) that $U(\cdot, \xi\theta) = \xi U(\cdot, \theta)$, for all $\xi > 0$.

Definition 3.4. The *individual index of TCF* for agent $i \in N$ is a map $\alpha^i : X \times \Theta^n \rightarrow \mathbb{R}_+$ defined by

$$\alpha^i(x_t, \theta_t) := -\frac{u'(s^i(x_t, \theta_t))}{u''(s^i(x_t, \theta_t))}.$$

Naturally, the *collective index of TCF*, $\alpha : X \times \Theta^n \rightarrow \mathbb{R}_+$ is defined in terms of $U(x_t, \theta_t)$ as

$$(3.4) \quad \alpha(x_t, \theta_t) := -\frac{\partial U(x_t, \theta_t)}{\partial x} \left[\frac{\partial^2 U(x_t, \theta_t)}{\partial x^2} \right]^{-1}.$$

A simple characterization of individual and collective indices of TCF is given next. A similar result is obtained in [11], for the case that x follows an exogenous process. Note that the result is independent of functional forms.

Proposition 3.5. Assume that $s \in \mathcal{S}$ is an optimal sharing rule. Then, for each $i \in N$ and for all $(x_t, \theta_t) \in X \times \Theta^n$, individual and aggregate indices of TCF satisfy the following properties

$$\frac{\partial s^i}{\partial x} = \frac{\alpha^i(x_t, \theta_t)}{\sum_j \alpha^j(x_t, \theta_t)}, \quad \text{and} \quad \alpha(x_t, \theta_t) = \sum_{i=1}^n \alpha^i(x_t, \theta_t).$$

Proof. See [11] and [20]. ■

In order to obtain the desired separability property for U , some additional results are borrowed from works on optimal risk sharing and syndicate theory. For instance, Amershi and Stoeckenius [1] established a sufficient condition for aggregation that requires separability, based on initial findings by Wilson [20]. The condition is related to a class of utility functions representing individual preferences that yield linear sharing rules.⁶ The same class of utility functions satisfies an analogous property in terms of TCF.

Definition 3.6. A utility function $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ satisfies *linear absolute tolerance for consumption fluctuations* (LTCF) if it has the form

$$(3.5) \quad u(x) = \frac{\gamma}{1-\gamma} \left[\left(\phi + \frac{\eta}{\gamma} x \right)^{1-\gamma} - 1 \right], \quad 0 < \gamma < +\infty, \gamma \neq 1,$$

with $\phi + (\eta/\gamma)x \geq 0$, $0 < \eta < +\infty$, and $\phi \in \mathbb{R}$. This family of parametric utility functions will be referred to as the *LTCF class*.

⁶In an environment with risk averse agents, interpreting $\theta_t \in \Theta^n$ as the realization of some random variable with a known distribution, if individual utility functions belong to the hyperbolic absolute risk aversion (HARA) class with identical cautiousness, optimal sharing rules are linear.

Remark 3.7. The LTCF class includes utility functions that are commonly used in the literature. For instance, $\phi = 0$ and $\eta = 1$ yields the power utility function. As $\gamma \rightarrow 1$, it converges to the logarithmic form $u(x) = \log(\phi + \eta x)$, with $\phi + \eta x > 0$. For $\phi = 1$, taking the limit $\gamma \rightarrow +\infty$ yields the exponential form $u(x) = 1 - \exp(-\eta x)$. Note that if $\phi < 0$, consumption has a lower bound, i.e., $x \geq -\phi\gamma/\eta$, then u takes the form of Stone-Geary preferences. The case $\phi \geq 0$ is compatible with $x \geq 0$.

The next result characterizes optimal sharing rules under LTCF preferences.

Proposition 3.8. *Assume that each agent has an instantaneous utility function u that belongs to the LTCF class. Further assume $k_0, \theta_0 > 0$ and that the solution to (S) is interior for every t . Then, the optimal sharing rule for each agent $i \in N$ has the form*

$$s^i(x_t, \theta_t) = a^i(\theta_t) x_t + b^i(\theta_t),$$

where $a^i(\theta_t) \geq 0$, $\sum_i a^i(\theta_t) = 1$, and $\sum_i b^i(\theta_t) = 0$, for all $\theta_t \in \Theta^n$.

Proof. For each i , the optimal sharing rule is given by

$$\tilde{s}(\theta_t^i, \lambda_t) = \frac{\gamma}{\eta} \left[\left(\frac{\eta \theta_t^i}{\lambda_t} \right)^{\frac{1}{\gamma}} - \phi \right], \quad t = 0, 1, \dots$$

Adding up over i yields aggregate consumption in terms of (θ_t, λ_t) ,

$$x_t = \sum_{i=1}^n \tilde{s}(\theta_t^i, \lambda_t) = \frac{\gamma}{\eta} \left[\left(\frac{\eta}{\lambda_t} \right)^{\frac{1}{\gamma}} \sum_{i=1}^n (\theta_t^i)^{\frac{1}{\gamma}} - \phi n \right], \quad t = 0, 1, \dots$$

Therefore,

$$\lambda_t = \left(\sum_{i=1}^n (\theta_t^i)^{\frac{1}{\gamma}} \right)^{\gamma} \eta \left(\phi n + \frac{\eta}{\gamma} x_t \right)^{-\gamma}, \quad t = 0, 1, \dots,$$

and after appropriate substitutions,

$$s^i(x_t, \theta_t) = \left[\frac{(\theta_t^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}} \right] x_t + \frac{\gamma \phi}{\eta} \left[\frac{(\theta_t^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}} n - 1 \right], \quad t = 0, 1, \dots$$

The sharing rule is linear in x_t , which implies

$$(3.6) \quad a^i(\theta_t) = \frac{(\theta_t^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}} \quad \text{and} \quad b^i(\theta_t) = \frac{\gamma \phi}{\eta} \left[\frac{(\theta_t^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}} n - 1 \right], \quad t = 0, 1, \dots,$$

as claimed. ■

Remark 3.9. Simple substitution shows that the aggregate instantaneous utility function has the form

$$(3.7) \quad U(x_t, \theta_t) = \frac{\gamma}{1-\gamma} \left[\left(\sum_i (\theta_t^i)^{\frac{1}{\gamma}} \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x_t \right)^{1-\gamma} - 1 \right],$$

which is clearly separable in x_t and θ_t (up to an additive constant). Note also that the aggregate TCF index is independent of θ_t and each individual TCF index is separable in (x_t, θ_t) , that is,

$$\hat{\alpha}_n(x_t) = \frac{1}{\eta} \left(\phi n + \frac{\eta}{\gamma} x_t \right) \quad \text{and} \quad \alpha^i(x_t, \theta_t) = a^i(\theta_t) \hat{\alpha}_n(x_t),$$

where $a^i(\theta_t)$ is defined in (3.6). Then, U can be alternatively expressed in terms of these quantities as

$$U(x_t, \theta_t) = \frac{\gamma}{1-\gamma} \left[\eta \sum_i (a^i(\theta_t))^{-\gamma} (\alpha_n(x_t))^{1-\gamma} - 1 \right].$$

3.2 Equivalence Results

This section presents several equivalent formulations of the Pareto problem (PP). For this, some of the previous results require additional specifications. Let $\mu : \Theta^n \rightarrow [0, 1]$ represent the aggregate discount rate defined as

$$(3.8) \quad \mu(\theta_t) := \sum_{i=1}^n \theta_t^i \delta^i, \quad t = 0, 1, \dots$$

Note that this function takes values on $[\delta_n, \delta_1] \subset [0, 1]$ for all $\theta_t \in \Theta^n$, it is strictly increasing in each argument and homogeneous of degree one.

Since μ is bounded on Θ^n , it is clear that the sequence of discount factors $\{\beta_t\}_{t=0}^\infty$ is bounded on the space $(0, 1)^\infty$. This, together with (2.18), implies that the program (RPF) is well-behaved, in a sense that will become clear below, and it satisfies a generalized Bellman equation.⁷ For $x_t > 0$ and θ_t in the interior of Θ^n , the Euler equation associated with (RPF) is given by

$$(3.9) \quad \frac{\partial U(x_t, \theta_t)}{\partial x} = \mu(\theta_t) \frac{\partial U(x_{t+1}, \theta_{t+1})}{\partial x} f'(k_{t+1}), \quad t = 0, 1, \dots,$$

and assuming LTCF preferences, it follows from (3.7) and (3.8) that

$$(3.10) \quad \left(\sum_i (\theta_t^i)^{\frac{1}{\gamma}} \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x_t \right)^{-\gamma} = \left(\sum_i \theta_t^i \delta^i \right) \left(\sum_i (\theta_{t+1}^i)^{\frac{1}{\gamma}} \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x_{t+1} \right)^{-\gamma} f'(k_{t+1}).$$

⁷The existence of a solution to functional equations arising in dynamic programming problems with generalized discounting has been shown under fairly general conditions in [3, 4] for the case of bounded returns, and in [15] for unbounded returns.

Given that condition (3.9) is necessary and sufficient for optimality, it is possible to define an aggregate utility function \hat{U} , which is independent of θ_t , and an effective discount rate $\hat{\mu}$ satisfying an equivalent condition, i.e.,

$$(3.11) \quad \hat{U}'(x_t) = \hat{\mu}(\theta_t) \hat{U}'(x_{t+1}) f'(k_{t+1}), \quad t = 0, 1, \dots$$

It is apparent from (3.10) that \hat{U} belongs to the LTCF class, as can be seen by substituting ϕ for ϕn and x_t^i for x_t in (3.5). Therefore,

$$(3.12) \quad \hat{U}(x_t) = \frac{\gamma}{1-\gamma} \left[\left(\phi n + \frac{\eta}{\gamma} x_t \right)^{1-\gamma} - 1 \right].$$

Collecting the terms that depend on θ_t and θ_{t+1} in (3.10) and using (2.18), the *effective aggregate discount rate* $\hat{\mu}(\theta)$ is defined as

$$(3.13) \quad \hat{\mu}(\theta_t) := \left[\frac{\sum_i (\theta_t^i \delta^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}} \right]^\gamma = \left[\sum_i a^i(\theta_t^i) (\delta^i)^{\frac{1}{\gamma}} \right]^\gamma.$$

As μ is a weighted arithmetic mean, the effective discount rate $\hat{\mu}$ is a *weighted generalized mean* (or *weighted power mean*) of the individual discount factors with weights $a^i(\theta_t^i)$, $i \in N$, and exponent γ . Only if individual preferences u are logarithmic, μ and $\hat{\mu}$ coincide.

Therefore, there is a dynamic program equivalent to (RPF). Since \hat{U} is differentiable, the solution satisfies an Euler equation equivalent to (3.10), which is given by (3.11). By Theorem 2.7, the equivalent program also satisfies (2.7). These results are summarized in the next proposition.

Proposition 3.10. *Assume that preferences are LTCF. Then, the following program is equivalent to (RPF)*

$$(RPF') \quad \begin{aligned} & \sup_{\mathbf{x} \in X(k_0), \mathbf{k} \in \Pi(k_0)} \sum_{t=0}^{\infty} \hat{\beta}_t \hat{U}(x_t) \\ & \text{s.t.} \quad x_t + k_{t+1} \leq f(k_t), \quad t = 0, 1, \dots, \\ & \quad \quad \hat{\beta}_{t+1} \leq \hat{\mu}(\theta_t) \hat{\beta}_t, \quad t = 0, 1, \dots, \\ & \quad \quad \theta_{t+1} = F(\theta_t), \quad t = 0, 1, \dots, \\ & \quad \quad k_0, \theta_0 > 0 \text{ given, } \hat{\beta}_0 = 1. \end{aligned}$$

Moreover, (RPF') satisfies the following Bellman equation

$$\hat{V}(k, \theta) = \sup_{y \in \Gamma(k)} \left[\hat{U}(f(k) - y) + \hat{\mu}(\theta) \hat{V}(y, F(\theta)) \right],$$

where \hat{U} and $\hat{\mu}$ are given by (3.12) and (3.13), respectively.

The last step to obtain preferences from (RPF') with the separable form (x, t) is completed as follows. It is clear from (2.18) that θ_t can be explicitly solved as a function of t and, in turn, the aggregate discount rate μ_t will depend on t directly and not through θ_t . This leads to a *nonstationary dynamic programming problem*, an approach that has been recently studied by Kamihigashi [16], and it involves an aggregate utility function $U : X \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ and a value function $V : K \times \mathbb{Z}_+ \rightarrow \mathbb{R}$ that depend directly on time.

For a given vector of initial utility weights $\theta_0 \in \Theta^n$, solving recursively for θ_t^i in (2.18)

$$(3.14) \quad \theta_t^i = \frac{\theta_0^i (\delta^i)^t}{\sum_i \theta_0^i (\delta^i)^t}, \quad t = 0, 1, \dots,$$

which immediately implies

$$(3.15) \quad \mu_t = \frac{\sum_i \theta_0^i (\delta^i)^{t+1}}{\sum_i \theta_0^i (\delta^i)^t}, \quad t = 0, 1, \dots$$

Next, set $\beta_0 = 1$ so each member of the sequence $\{\beta_t\}_{t=0}^\infty$ of discount factors is given by

$$(3.16) \quad \beta_t = \prod_{s=0}^{t-1} \mu_s = \sum_{i=1}^n \theta_0^i (\delta^i)^t, \quad \text{for all } t \geq 1.$$

Suppose that $\theta_0^i > 0$ for each $i \in N$. The following nonstationary formulation (NSF) of the Pareto problem is equivalent to (RPF) and has the form

$$(NSF) \quad \begin{aligned} & \sup_{x \in X(k_0), k \in \Pi(k_0)} \liminf_{T \rightarrow \infty} \sum_{t=0}^T \beta_t U_t(x_t) \\ & \text{s.t.} \quad x_t + k_{t+1} \leq f(k_t), \quad t = 0, 1, \dots, \\ & \quad k_0 > 0 \text{ given.} \end{aligned}$$

In addition, the value function V_t associated to (NSF) satisfies the (modified) Bellman equation

$$V_t(k_t) = \sup_{k_{t+1} \in \hat{\Gamma}(k_t)} \left[U_t(f(k_t) - k_{t+1}) + \beta_{t+1} V_{t+1}(k_{t+1}) \right],$$

where β_t is given by (3.16) for each t , and $\hat{\Gamma}(k_t)$ is defined as

$$(3.17) \quad \hat{\Gamma}(k_t) := \left\{ k_{t+1} \in \Gamma(k_t) : U_t(f(k_t) - k_{t+1}) > -\infty \right\}.$$

This result is a straightforward application of Theorem 1 in [16].⁸

⁸It can also be shown that a Bellman equation holds in a strict sense if and only if there is no $k_{t+1} \in \Gamma(k_t)$ such that $U_t(f(k_t) - k_{t+1}) = -\infty$ and $V_{t+1}(k_{t+1}) = +\infty$ simultaneously. See [16, Theorem 2].

Note that the previous formulation does not assume specific functional forms. Under LTCF preferences, instantaneous utility for the NSF is obtained substituting (3.14) and (3.15) into (3.7), which yields

$$U_t(x) = \frac{\gamma}{1-\gamma} \left[\frac{\left(\sum_i (\theta_0^i (\delta^i)^t)^{\frac{1}{\gamma}} \right)^\gamma}{\sum_i \theta_0^i (\delta^i)^t} \left(\phi n + \frac{\eta}{\gamma} x \right)^{1-\gamma} - 1 \right].$$

This function satisfies the following asymptotic properties that considerably simplify the formulation of the nonstationary program,

$$U_0(x) := \lim_{t \rightarrow 0^+} U_t(x) = \frac{\gamma}{1-\gamma} \left[\left(\sum_i (\theta_0^i)^{\frac{1}{\gamma}} \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x \right)^{1-\gamma} - 1 \right], \quad \text{for all } x \in X,$$

and

$$\lim_{t \rightarrow +\infty} U_t(x) = \frac{\gamma}{1-\gamma} \left[\left(\phi n + \frac{\eta}{\gamma} x \right)^{1-\gamma} - 1 \right] = U(x), \quad \text{for all } x \in X.$$

Moreover, the sequence $\{U_t\}_{t=0}^\infty$ is strictly decreasing if $0 < \gamma \leq 1$ (resp. strictly increasing if $1 < \gamma < \infty$) for any $x \in X$, and takes values on the closed interval $[U(x), U_0(x)]$ (resp. $[U_0(x), U(x)]$).

Let $\hat{\theta}_0^i$ denote the *effective utility weights* for each agent i , defined as

$$\hat{\theta}_0^i := \frac{(\theta_0^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_0^i)^{\frac{1}{\gamma}}}, \quad i = 1, \dots, n,$$

and note that $0 \leq \hat{\theta}_0^i \leq 1$ and $\sum_i \hat{\theta}_0^i = 1$. For convenience and ease of notation, define also the *effective individual discount factors* as $\hat{\delta}^i := (\delta^i)^{\frac{1}{\gamma}}$, so the *effective discount factor* $\hat{\beta}_t$ associated with (NSF) is given by

$$(3.18) \quad \hat{\beta}_t = \left(\sum_{i=1}^n \hat{\theta}_0^i (\hat{\delta}^i)^t \right)^\gamma, \quad t = 0, 1, \dots$$

In a similar way as in the previous case, the Euler equation associated with (NSF), is equivalent to (3.10) with instantaneous utility $\hat{U}(x)$ and effective discount factor $\hat{\beta}_t$,

$$\hat{\beta}_t \hat{U}'(x_t) = \hat{\beta}_{t+1} \hat{U}'(x_{t+1}) f'(k_{t+1}),$$

therefore

$$\left(\sum_{i=1}^n \hat{\theta}_0^i (\hat{\delta}^i)^t \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x_t \right)^{-\gamma} = \left(\sum_{i=1}^n \hat{\theta}_0^i (\hat{\delta}^i)^{t+1} \right)^\gamma \left(\phi n + \frac{\eta}{\gamma} x_{t+1} \right)^{-\gamma} f'(k_{t+1}).$$

The next proposition, analogous to Proposition 3.10, summarizes the results for the NSF.

Proposition 3.11. *Assume that preferences are LTCF. Then, there exists an equivalent program to (NSF) which has the form*

$$\begin{aligned}
 (\text{NSF}') \quad & \sup_{\mathbf{x} \in X(k_0), \mathbf{k} \in \Pi(k_0)} \sum_{t=0}^{\infty} \hat{\beta}_t \hat{U}(x_t) \\
 \text{s.t.} \quad & x_t + k_{t+1} \leq f(k_t), \quad t = 0, 1, \dots, \\
 & k_0 > 0 \text{ given,}
 \end{aligned}$$

and satisfies the (modified) Bellman equation

$$\hat{V}_t(k_t) = \sup_{k_{t+1} \in \hat{\Gamma}(k_t)} \left[\hat{U}(f(k_t) - k_{t+1}) + \hat{\beta}_{t+1} \hat{V}_{t+1}(k_{t+1}) \right], \quad t = 0, 1, \dots,$$

where $\hat{\Gamma}$ has been defined in (3.17) and $\hat{\beta}_t$, $t = 0, 1, \dots$, is given by (3.18).

Having completed the construction of collective preferences of the sought form for U , the following section offers a characterization of these preferences.

4 Discussion

Several properties of collective intertemporal utility functions are discussed next. First, preferences are characterized in terms of impatience. In particular, it is shown that they satisfy decreasing marginal impatience. Second, the properties of stationarity, time invariance, and time consistency introduced in Section 1 are verified. To conclude, a comparison of the model developed in the previous section is made with a formulation of the Pareto problem under constant utility weights and exogenous consumption streams.

4.1 Collective impatience

Definition 4.1. Given an aggregate instantaneous utility function $U(x_t, \theta_t)$ and a discount factor β_t , the marginal rate of intertemporal substitution between x_t and x_{t+1} is defined as

$$\mathcal{M}[(x_t, x_{t+1}); (\theta_t, \theta_{t+1})] := \beta_t \frac{\partial U(x_t, \theta_t)}{\partial x} \left[\beta_{t+1} \frac{\partial U(x_{t+1}, \theta_{t+1})}{\partial x} \right]^{-1}.$$

Impatience is typically measured as the pure rate of time preference $\hat{\rho}_t$, which in turn is defined as the marginal rate of intertemporal substitution when $x_t = x_{t+1} = x$.

If the function U belongs to the LTCF class, as in (3.7), it follows that

$$(1 + \hat{\rho}_t) := \frac{\left(\sum_i (\theta_t^i)^{\frac{1}{\gamma}}\right)^\gamma \left(\phi n + \frac{n}{\gamma} x\right)^{-\gamma}}{\left(\sum_i \delta^i \theta_t^i\right) \left(\sum_i (\theta_{t+1}^i)^{\frac{1}{\gamma}}\right)^\gamma \left(\phi n + \frac{n}{\gamma} x\right)^{-\gamma}} = \left[\frac{\sum_i (\theta_t^i)^{\frac{1}{\gamma}}}{\sum_i (\theta_t^i \delta^i)^{\frac{1}{\gamma}}} \right]^\gamma,$$

and the discount rate implied by this expression is exactly $\hat{\rho}_t = 1/\hat{\mu}(\theta_t) - 1$, where $\hat{\mu}(\cdot)$ is given by (3.13). But this yields the same effective discount factor as in (3.18). Therefore, the above expression offers an alternative way to obtain the effective aggregate discount factor which is independent of the optimization process.

The following lemma is needed for future reference, but it is also an interesting result on its own, since it characterizes the behavior of aggregate discount factors as a function of time.

Lemma 4.2. *Assume that β_t is defined as in (3.16). Then, for every $0 \leq t \leq t'$ and $0 \leq \tau \leq \tau'$,*⁹

$$\frac{\beta_{t+\tau}}{\beta_{t+\tau'}} - \frac{\beta_{t'+\tau}}{\beta_{t'+\tau'}} \geq 0,$$

with strict inequality if $t < t'$ and $\tau < \tau'$.

Proof. Let $\Delta t, \Delta \tau$ be nonnegative integers such that $t' = t + \Delta t$ and $\tau' = \tau + \Delta \tau$. Then, the sign of $\beta_{t+\tau}/\beta_{t+\tau'} - \beta_{t'+\tau}/\beta_{t'+\tau'}$ is the same as

$$\left[\sum_{i=1}^n \theta_0^i (\delta^i)^{t+\tau} \right] \left[\sum_{j=1}^n \theta_0^j (\delta^j)^{t+\Delta t+\tau+\Delta \tau} \right] - \left[\sum_{i=1}^n \theta_0^i (\delta^i)^{t+\tau+\Delta \tau} \right] \left[\sum_{j=1}^n \theta_0^j (\delta^j)^{t+\Delta t+\tau} \right].$$

After expanding both products in this expression, it is easy to see that all factors with $i = j$ cancel each other out. Collecting the remaining terms yields

$$\sum_{\substack{i,j=1 \\ i < j}}^n [\theta_0^i (\delta^i)^{t+\tau}] [\theta_0^j (\delta^j)^{t+\tau}] [(\delta^i)^{\Delta t} - (\delta^j)^{\Delta t}] [(\delta^i)^{\Delta \tau} - (\delta^j)^{\Delta \tau}].$$

Note that if either Δt or $\Delta \tau$ is zero, the above sum vanishes. Otherwise, provided that $\Delta t > 0$ and $\Delta \tau > 0$, it follows from hypothesis that $\delta^i - \delta^j > 0$ for at least one pair i, j such that $i > j$. Hence the sum must be strictly positive. This completes the proof. ■

Clearly, β_t is decreasing in t which reflects the presence of impatience at the aggregate level. In addition, Lemma 4.2 establishes that collective preferences, as constructed in the previous sections, satisfy the property of *diminishing marginal impatience*, in a sense consistent with a discrete-time formulation.

⁹This can be assumed without loss of generality, given that it is always possible to relabel t, t' and τ, τ' in order to satisfy the required inequalities.

4.2 Stationarity, time invariance, and time consistency

In terms of the three axioms introduced in [12], it is shown here that the model of collective choice developed in Section 3 yields collective preferences that are both nonstationary and time-dependent. At the same time, and perhaps surprisingly, these preferences satisfy time consistency.

Before presenting the main result of this section, note that discount factors β_t (or, equivalently, $\hat{\beta}_t$) transform period t values into period 0 units. Given the multiplicative nature of discount factors derived from recursive preferences, it is easy to see that for any $t, \tau \geq 0$, discounting from $(t + \tau)$ to t is equivalent to multiplication by the factor $\beta_{t+\tau}/\beta_t$.

Theorem 4.3. *Suppose that a sequence of collective preference relations over (x, t) is represented by a separable utility function of the form $\hat{\beta}_t \hat{U}(x)$, where $\hat{\beta}_t$ and \hat{U} are given by (3.18) and (3.12), respectively. Furthermore, assume $\theta_0^i > 0$ for each $i \in N$. Then, collective time preferences satisfy:*

- \neg (A1) nonstationarity,
- \neg (A2) time dependency,
- (A3) time consistency.

Proof. First, note that the conclusion of Lemma 4.2 remains valid if β_t is replaced by $\hat{\beta}_t$. The reason is because substituting θ_0^i with $\hat{\theta}_0^i$ and δ^i with $\hat{\delta}^i$ in the definition of β_t yields $\sum_i \hat{\theta}_0^i (\hat{\delta}^i)^t$. It is easy to see from (3.18) that $\hat{\beta}_t$ is an order-preserving transformation of β_t with respect to t . Then, the Lemma also holds for this monotonic transformation.

Now assume that $b, c \in X$, $0 \leq t < t'$ and $0 \leq \tau < \tau'$. From (A1), one of the indifference conditions for stationarity (on the left-hand side) is equivalent to

$$(4.1) \quad \frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_t} \hat{U}(b) = \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_t} \hat{U}(c).$$

Hence, the following equality (the right-hand side) must hold,

$$\frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_t} \frac{\hat{\beta}_{t'+\tau}}{\hat{\beta}_{t+\tau}} \hat{U}(b) = \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_t} \frac{\hat{\beta}_{t'+\tau'}}{\hat{\beta}_{t+\tau'}} \hat{U}(c),$$

which is equivalent to

$$(4.2) \quad \frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_{t+\tau'}} - \frac{\hat{\beta}_{t'+\tau}}{\hat{\beta}_{t'+\tau'}} = 0.$$

To verify (A2), suppose that (4.1) holds. Time invariance requires that, from the perspective of t' preferences, applying the same time delays (τ and τ') to each alternative

preserves indifference, that is,

$$(4.3) \quad \frac{\hat{\beta}_{t'+\tau}}{\hat{\beta}_t'} \hat{U}(b) = \frac{\hat{\beta}_{t'+\tau'}}{\hat{\beta}_t'} \hat{U}(c).$$

Combine (4.1) and (4.3) to obtain

$$\frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_{t'+\tau}} - \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_{t'+\tau'}} = \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_{t'+\tau}} \left(\frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_{t+\tau'}} - \frac{\hat{\beta}_{t'+\tau}}{\hat{\beta}_{t'+\tau'}} \right) = 0,$$

which reduces to (4.2). In other words, the same condition must be satisfied for (A1) and (A2) to hold. But Lemma 4.2 implies that if $t < t'$ and $\tau < \tau'$, then

$$\frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_{t+\tau'}} - \frac{\hat{\beta}_{t'+\tau}}{\hat{\beta}_{t'+\tau'}} > 0,$$

which contradicts (4.2). This proves $\neg(\text{A1})$ and $\neg(\text{A2})$.

From (A3), time consistency implies that the following two conditions are satisfied simultaneously,

$$\frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_t} \hat{U}(b) = \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_t} \hat{U}(c) \quad \text{and} \quad \frac{\hat{\beta}_{t+\tau}}{\hat{\beta}_{t'}} \hat{U}(b) = \frac{\hat{\beta}_{t+\tau'}}{\hat{\beta}_{t'}} \hat{U}(c).$$

Since $\hat{\beta}_t$ and $\hat{\beta}_{t'}$ are strictly positive by hypothesis, it immediately follows that collective preferences are time consistent and the proof is complete. \blacksquare

Note that the proof of Theorem 4.3 captures an interesting aspect of the analysis carried out in [12]. Given that any two properties imply the third, if one property is satisfied, say, time consistency, and another property is not, e.g., stationarity, the third property must not be satisfied. This is the reason why the conditions for stationarity and time invariance are identical.

4.3 Heterogeneous discounting with constant utility weights

To close this section, the results obtained from the current framework will be compared with [13, 14] to offer some insights into their results and evaluate possible extensions to this paper. The authors show that under heterogeneous discounting, collective preferences that satisfy Pareto optimality must be either time inconsistent or dictatorial. But they assume constant utility weights and a common consumption stream that is exogenous. Consequently, key instruments to resolve intertemporal conflicts are missing.

For simplicity, suppose that u is bounded on \mathbb{R}_+ . Let $\bar{\theta} \in \Theta^n$ be a vector of constant (time homogeneous) utility weights, and add to (PP) the restriction $\theta_t^i = \theta_{t+1}^i = \bar{\theta}^i \in \bar{\theta}$ for all i , and for all t . Let $J(\cdot, \bar{\theta})$ denote the corresponding value function. Then, the

right-hand side of (2.8) is replaced by

$$\sup_{(\hat{x}, y, z, \lambda) \in \Phi} \inf_{\mu \geq 0} \left\{ \sum_i \bar{\theta}^i [u(x^i) - \lambda \sum_i x^i] + \lambda [f(k) - y] + \sum_i \bar{\theta}^i (\delta^i - \mu) z^i + \mu J(y, \bar{\theta}) \right\}.$$

Given that the choice of \hat{x} can still be separated from the investment decision and the resource restriction holds with equality, following similar arguments as in the proof of Theorem 2.7, the above problem is reduced to

$$(4.4) \quad \sup_{z \in \mathcal{U}} \sup_{y \in \Gamma(k)} \inf_{\mu \geq 0} \left\{ U(f(k) - y, \bar{\theta}) + [\sum_i \bar{\theta}^i \delta^i z^i - \mu \sum_i \bar{\theta}^i z^i] + \mu J(y, \bar{\theta}) \right\}.$$

By hypothesis, each z^i must be chosen from some interval $Z^i := [0, z_m^i]$ with $z_m^i > 0$.

Consider, for instance, an “egalitarian solution” to (4.4) in which $z^i = z > 0$ and $z \in \bigcap_i Z^i$ for all $i \in N$. Hence, $J(y, \bar{\theta}) = z$. This implies that

$$J(k, \bar{\theta}) = \sup_{y \in \Gamma(k)} \left\{ U(f(k) - y, \bar{\theta}) + \delta J(y, \bar{\theta}) \right\},$$

so the aggregate discount factor is the weighted average $\delta := (\sum_i \bar{\theta}^i \delta^i)$ of all individual discount factors. This formulation can be interpreted as the dynamic program of a fictional “representative agent” with an average discount factor and time consistent preferences. But these preferences will not satisfy unanimity, unless all the $\bar{\theta}^i$ are identical (i.e., equal to $(1/n)$ for each i).

Another possibility is to consider an “efficient solution” that consists in setting μ so that $\bar{\theta}^i (\delta^i - \mu) = 0$, the first-order condition from differentiating the objective with respect to each z^i . However, this condition may be satisfied only for a single $i \in N$. Which value of δ^i should be chosen? Suppose that $\mu = \delta^1$. It follows that $z^j = 0$ for all $j > 1$, hence the problem reduces to

$$(4.5) \quad J(k, \bar{\theta}) = \sup_{y \in \Gamma(k)} \left\{ U(f(k) - y, \bar{\theta}) + \delta^1 J(y, \bar{\theta}) \right\}.$$

But this is equivalent to finding the optimum for $i = 1$ and leaving the other agents with zero utility, thus collective preferences will be dictatorial. Assume that $\mu = \delta^i$ for some i in $N \setminus \{1\}$. In that case, $z_k = 0$ for all $k > i$, so the problem has the form

$$(4.6) \quad \sup_{y \in \Gamma(k)} \sup_{(z^1, \dots, z^i) \in Z^1 \times \dots \times Z^i} \left\{ U(f(k) - y, \bar{\theta}) + \sum_{j < i} \bar{\theta}^j (\delta^j - \delta^i) z^j + \delta^i J(y, \bar{\theta}) \right\}.$$

Note that the marginal contribution of each agent $1 \leq j < i$ to the aggregate continuation value is precisely $\bar{\theta}^j (\delta^j - \delta^i)$, hence the choice of (z^1, \dots, z^i) can be formulated as a linear

programming problem, i.e.,

$$\sup_{(z^1, \dots, z^i) \in Z^1 \times \dots \times Z^i} \sum_{j < i} \bar{\theta}^j (\delta^j - \delta^i) z^j.$$

If the solution lies on a vertex of the convex polytope that describes the feasible region, then a single agent gets nonzero utility. If it lies on an edge or a face of the polytope, two or more agents obtain positive continuation utilities, but they are linear combinations of each other. In any case, the solution of this auxiliary problem will be dictatorial. Call the solution of the above problem $\bar{z}_j \in Z^j$ for each $1 \leq j < i$. Then (4.6) can be written as

$$(4.7) \quad \sup_{y \in \Gamma(k)} \left[U(f(k) - y, \bar{\theta}) + \delta^i J(y, \bar{\theta}) \right] + \sum_{j < i} \bar{\theta}^j \delta^j \bar{z}^j,$$

so the choice of y at the margin is determined by δ^i , whereas each of the $\bar{\theta}^j \delta^j \bar{z}^j$ terms works effectively as a lump-sum transfer to those agents with a higher level of patience than i .

It is easy to show that at any period t , an optimal sharing rule \bar{x}_t^j must satisfy

$$\left(\frac{\delta^i}{\delta^j} \right)^t = \frac{\bar{\theta}^j u'(\bar{x}_t^j)}{\bar{\theta}^i u'(\bar{x}_t^i)}, \quad 1 \leq j < i; \quad t = 0, 1, \dots,$$

hence as $t \rightarrow +\infty$, the ratio of marginal utilities $u'(\bar{x}_t^j)/u'(\bar{x}_t^i) \rightarrow 0$, so consumption of each agent j should grow faster than consumption of i . The above condition immediately implies

$$\frac{u'(\bar{x}_t^i)}{\delta^i u'(\bar{x}_{t+1}^i)} = \frac{u'(\bar{x}_t^j)}{\delta^j u'(\bar{x}_{t+1}^j)}, \quad 1 \leq j < i; \quad t = 0, 1, \dots$$

At the same time, the evolution of aggregate variables is determined by the following Euler equation

$$\frac{\partial U(\bar{x}_t)}{\partial x} = \delta^i \frac{\partial U(\bar{x}_{t+1})}{\partial x} f'(\bar{y}_t), \quad t = 0, 1, \dots,$$

where $\bar{y}_t \in \Gamma(k_t)$ solves (4.7) and feasibility requires

$$\bar{x}_t = \sum_{j < i} \bar{x}_t^j + \bar{x}_t^i = f(k_t) - \bar{y}_t.$$

But a program with such characteristics is clearly time inconsistent. To summarize, if the utility weights are restricted to be constant over time and the consumption-savings choice is endogenous, an interior solution for all agents may be difficult (or even impossible) to achieve. In addition, the problems of time inconsistency and dictatorial preferences may become interrelated and more complex in nature.

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